$\sim$ midterm: differentiability of multi-variable function. $\to f: \mathbb{R}^n \to \mathbb{R}$

midterm $\sim$: " " vector-valued multi-variable function

$f: \mathbb{R}^n \to \mathbb{R}^m$

differentiability = linear approximation

multi-variable function; $L(\vec{x}) = f(\vec{a}) + \underbrace{\nabla f(\vec{a})}_{\text{gradient of } f} \cdot (\vec{x} - \vec{a})$

vector-valued multivariable function: $\hookrightarrow$ Jacobian matrix

<u>Recall</u> matrix multiplication

A: $m \times n$ matrix, B: $n \times k$ matrix

$\Rightarrow AB$ $m \times k$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix} = \begin{pmatrix} ae+bh & af+bi & ag+bj \\ ce+dh & cf+di & cg+dj \end{pmatrix}$$

$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{pmatrix}$$

m rows, columns

If $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | \\ \vec{b} \\ | \end{pmatrix} \in \mathbb{R}^n$ (written as a column vector)

$$A\vec{b} = \begin{pmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{pmatrix} \begin{pmatrix} | \\ \vec{b} \\ | \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + \cdots + a_{1n}b_n \\ \vdots \\ a_{m1}b_1 + \cdots + a_{mn}b_n \end{pmatrix}$$

$$= \underset{m \times 1}{\begin{pmatrix} \vec{a}_1 \cdot \vec{b} \\ \vdots \\ \vec{a}_m \cdot \vec{b} \end{pmatrix}} \in \mathbb{R}^m$$

similarly,

$$\underset{1 \times n}{\begin{pmatrix} - & \vec{a} & - \end{pmatrix}} \underset{n \times k}{\begin{pmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{pmatrix}} = \underset{1 \times k}{(\vec{a} \cdot \vec{b}_1, \cdots, \vec{a} \cdot \vec{b}_k)}$$

$$\underset{m \times n}{A} \underset{n \times k}{B} = A \begin{pmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ A\vec{b}_1 & \cdots & A\vec{b}_k \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{pmatrix} B = \begin{pmatrix} - & \vec{a}_1 B & - \\ & \vdots & \\ - & \vec{a}_m B & - \end{pmatrix}$$

eg $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$

$$AB = \begin{pmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$$

$$A\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 5+16 \\ 15+32 \end{pmatrix} = \begin{pmatrix} 21 \\ 47 \end{pmatrix}$$

$$A\begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 24 \\ 54 \end{pmatrix} \quad A\begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} 27 \\ 61 \end{pmatrix}$$

$$(1 \ \ 2)B = (21 \ \ 24 \ \ 27)$$
$$(3 \ \ 4)B = (47 \ \ 54 \ \ 61)$$

Vector-valued functions

$$\vec{f} : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^m$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$$ vector-valued.

Each $f_i(\vec{x})$ is a multi-variable function.

Suppose $\frac{\partial f_i}{\partial x_j}(\vec{a})$ for each i, j.

For each $1 \le i \le m$

$$\underset{1\times 1}{f_i(\vec{x})} = \underset{1\times 1}{f_i(\vec{a})} + \underset{1\times n}{\nabla f_i(\vec{a})} \cdot \underset{n\times 1}{(\vec{x}-\vec{a})} + \underset{1\times 1}{\epsilon_i(\vec{x})}$$

regard as $1\times n$ row vector

regarded as $n\times 1$ column vector

$$\begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} = \begin{pmatrix} f_1(\vec{a}) \\ \vdots \\ f_m(\vec{a}) \end{pmatrix} + \underbrace{\begin{pmatrix} - \nabla f_1(\vec{a}) - \\ \vdots \\ - \nabla f_m(\vec{a}) - \end{pmatrix}}_{m\times n \text{ matrix}} \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix} + \begin{pmatrix} \epsilon_1(\vec{x}) \\ \vdots \\ \epsilon_m(\vec{x}) \end{pmatrix}$$

Def ① The <u>Jacobian matrix</u> of $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ at $\vec{a}$ is

$$D\vec{f}(a) = \begin{pmatrix} - \nabla f_1(\vec{a}) - \\ \vdots \\ - \nabla f_m(\vec{a}) - \end{pmatrix} \quad (m\times n \text{ matrix})$$

② The <u>linearization of $\vec{f}$</u> at $\vec{a}$ to be

$$\vec{L}(\vec{x}) = \vec{f}(\vec{a}) + D\vec{f}(\vec{a}) \cdot (\vec{x}-\vec{a})$$

③ $\vec{f}$ is <u>differentiable</u> at $\vec{a}$ if Error term

$\vec{\varepsilon}(\vec{x}) = \vec{f}(\vec{x}) - \vec{L}(\vec{x})$ satisfies

$$\lim_{\vec{x}\to\vec{a}} \frac{\|\vec{\varepsilon}(\vec{x})\|}{\|\vec{x}-\vec{a}\|} = 0 .$$

<u>Rmk</u> ① $[D\vec{f}(\vec{a})]_{ij} = \dfrac{\partial f_i}{\partial x_j}(\vec{a})$

② $$\underset{m\times 1}{\vec{f}(\vec{x})} = \underset{m\times 1}{\vec{f}(\vec{a})} + \underbrace{\underset{m\times n}{D\vec{f}(\vec{a})}\cdot\underset{n\times 1}{(\vec{x}-\vec{a})}}_{m\times 1} + \underset{m\times 1}{\vec{\varepsilon}(\vec{x})}$$

③ If f is real-valued (i.e. $m=1$)

$D\vec{f}(\vec{a}) = \nabla f(\vec{a})$

④ $\|\vec{\varepsilon}(\vec{x})\|$: length in $\mathbb{R}^m$

$\|\vec{x}-\vec{a}\|$: length in $\mathbb{R}^n$.

⑤ $$\lim_{\vec{x}\to\vec{a}} \frac{\|\vec{\varepsilon}(\vec{x})\|}{\|\vec{x}-\vec{a}\|} = 0 \iff \lim_{\vec{x}\to\vec{a}} \frac{\varepsilon_i(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0$$ for $i=1,\cdots,m$

$\therefore \vec{f}$ is differentiable at $\vec{a} \iff f_i$ is differentiable at $\vec{a}$ for $i=1,\cdots,m$

Approximation $\vec f(\vec x) \approx \vec L(\vec x) = \vec f(\vec a) + D\vec f(\vec a)\cdot(\vec x - \vec a)$

$$\Rightarrow \underbrace{\vec f(\vec x) - \vec f(\vec a)}_{\Delta \vec f \,=\, \text{change of } \vec f} \approx D\vec f(\vec a)\cdot\underbrace{(\vec x - \vec a)}_{\Delta \vec x \,=\, \text{change of } \vec x.}$$

One can consider $D\vec f(\vec a)$ as a linear map

$$D\vec f(\vec a) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\Delta \vec x \longmapsto D\vec f(\vec a)\,\Delta \vec x.$$

eg $\vec f(x,y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (y+1)\ln x \\ x^2 - \sin y + 1 \end{pmatrix} : \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$

① What is $D\vec f(1,0)$?

$\nabla f_1 = \left(\frac{y+1}{x} \quad \ln x\right)$, $\nabla f_2 = (2x \quad -\cos y)$

$$\therefore D\vec f(x,y) = \begin{pmatrix} \frac{y+1}{x} & \ln x \\ 2x & -\cos y \end{pmatrix}$$

$$D\vec f(1,0) = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

② linearization of $\vec f$ at $(1,0)$ $\quad \vec x - \vec a$

$$\vec L(x,y) = \vec f(1,0) + D\vec f(1,0)\cdot\left(\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}1&0\\2&-1\end{pmatrix}\begin{pmatrix}x-1\\y\end{pmatrix}$$

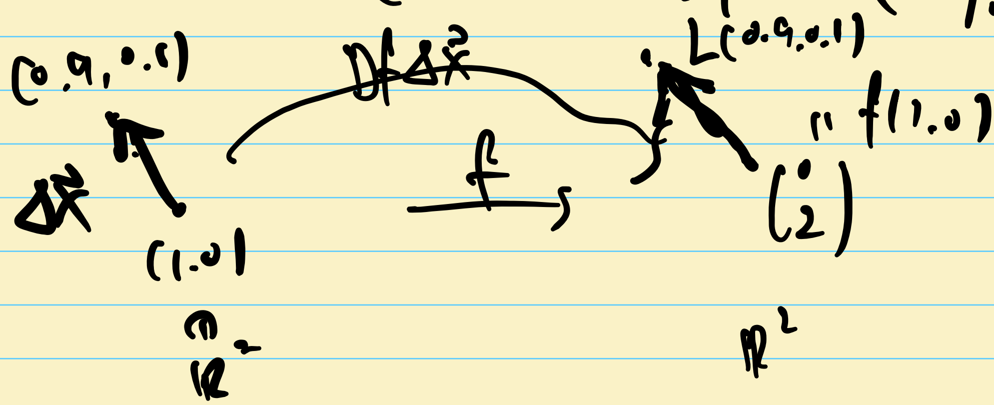
③ approximation of $\vec f(0.9, 0.1)$

$$\vec f(0.9, 0.1) \approx \vec L(0.9, 0.1)$$

$$= \begin{pmatrix}0\\2\end{pmatrix} + \underbrace{\begin{pmatrix}1&0\\2&-1\end{pmatrix}\underbrace{\begin{pmatrix}0.9-1\\0.1\end{pmatrix}}_{\Delta\vec x}}_{D\vec f(\vec a)\Delta\vec x}$$

$$= \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-0.1\\-0.2-0.1\end{pmatrix}$$

$$= \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-0.1\\-0.3\end{pmatrix} = \begin{pmatrix}-0.1\\1.7\end{pmatrix}$$



Rmk Total differential can be written in matrix form

$\vec{f} : \Omega (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^m$

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \qquad d\vec{f} = \begin{pmatrix} df_1 \\ \vdots \\ df_m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} dx_j \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial f_m}{\partial x_j} dx_j \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}}_{D\vec{f}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

$$= D\vec{f}(\vec{a}) \cdot d\vec{x} .$$

Chain rule: differential of composition of two functions.

__Recall__ in one-variable

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R} \qquad (g\circ f)'(x)$$

$$\underbrace{\qquad\qquad}_{g\circ f} \qquad = g'(f(x)) \cdot f'(x)$$

$w = g(u)$ $\qquad \frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx}$

$u = f(x)$

eg $w = 2u+1$ $\qquad \frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx}$

$u = x^2$

$= 2 \cdot 2x = 4x.$

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R} \qquad \nabla(g\circ f)(\vec{a})$$

$$\underbrace{\qquad\qquad}_{g\circ f} \qquad = g'(f(\vec{a})) \cdot \nabla f(\vec{a})$$

For vector-valued multi-variable functions.

__Thm__ (Chain rule)

Let $\vec{f}: \Omega_1 (\subseteq \mathbb{R}^k) \to \mathbb{R}^n$, $\vec{g}: \Omega_2 (\subseteq \mathbb{R}^n) \to \mathbb{R}^m$

Suppose $\vec{f}$ is differentiable at $\vec{a} \in \Omega_1$

" $\vec{g}$ " " $\vec{b} = f(\vec{a}) \in \Omega_2$

$$\begin{tikzcd} \vec{a} \in \mathbb{R}^k \arrow[r, "\vec{f}"] \arrow[rr, bend right, "\vec{g}\circ\vec{f}"'] & \mathbb{R}^n \arrow[r, "\vec{g}"] & \mathbb{R}^m \end{tikzcd}$$

$\vec{g}\circ\vec{f}$ is differentiable at $\vec{a}$ and

$$\underset{m\times k}{D(\vec{g}\circ\vec{f})(\vec{a})} = \underset{m\times n}{D\vec{g}(\vec{f}(\vec{a}))} \cdot \underset{n\times k}{D\vec{f}(\vec{a})}$$

Rmk For simplicity, we omit $\rightarrow$ for vectors from now on. $\vec{f} = f$, $\vec{x} = x$ etc.

eg $f : \mathbb{R} \to \mathbb{R}^2$ $\quad f(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$g : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(u,v) = \begin{pmatrix} 2uv \\ u^2 - v^2 \end{pmatrix}$

$D(g\circ f)(\theta)$ $\quad g\circ f : \mathbb{R} \to \mathbb{R}^2$

(sol) direct computation.

$$(g\circ f)(\theta) = g(\cos\theta, \sin\theta) = \begin{pmatrix} 2\cos\theta\sin\theta \\ \cos^2\theta - \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin 2\theta \\ \cos 2\theta \end{pmatrix}$$

$$\therefore D(g \circ f)(\theta) = \begin{pmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{pmatrix}$$

apply chain rule,

$$Df(\theta) = \begin{pmatrix} f_1'(\theta) \\ f_2'(\theta) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$Dg(u,v) = \begin{pmatrix} 2v & 2u \\ 2u & -2v \end{pmatrix}$$

$$D(g \circ f)(\theta) = Dg(f(\theta)) \cdot Df(\theta)$$

$$= \begin{pmatrix} 2\sin\theta & 2\cos\theta \\ 2\cos\theta & -2\sin\theta \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -2\sin^2\theta + 2\cos^2\theta \\ -2\cos\theta\sin\theta - 2\cos\theta\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{pmatrix}$$

eg $\quad f(x,y) = \begin{pmatrix} x^2 \\ 3xy \\ x+y^2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3 . \quad g(u,v,w) = \dfrac{uw}{v} : \mathbb{R}^3 \to \mathbb{R}$

$\dfrac{\partial (g\circ f)}{\partial x}(1,1)$?

(sol) $Dg = \nabla g = \left(\dfrac{w}{v}, -\dfrac{uw}{v^2}, \dfrac{u}{v} \right)$

$Df = \begin{pmatrix} 2x & 0 \\ 3y & 3x \\ 1 & 2y \end{pmatrix} \qquad f(1,1) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\therefore D(g\circ f)(1,1) = Dg(f(1,1)) \cdot Df(1,1)$

$= \left(\dfrac{2}{3}, -\dfrac{2}{9}, \dfrac{1}{2} \right) \cdot \begin{pmatrix} 2 & 0 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}$

$= (1 \;\; 0)$

$\dfrac{\partial (g\circ f)}{\partial x}(1,1) \qquad \therefore \dfrac{\partial (g\circ f)}{\partial x}(1,1) = 1 \,!$

Rmk In classical notation, we write $\frac{\partial g}{\partial x}$ for $\frac{\partial (g\circ f)}{\partial x}$. Using classical notation, chain rule can be written as (in the previous example)

$$\begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial g}{\partial v}\cdot\dfrac{\partial v}{\partial x} + \dfrac{\partial g}{\partial w}\cdot\dfrac{\partial w}{\partial x} \\[2ex] \dfrac{\partial g}{\partial y} = \dfrac{\partial g}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial g}{\partial v}\cdot\dfrac{\partial v}{\partial y} + \dfrac{\partial g}{\partial w}\cdot\dfrac{\partial w}{\partial y} \end{cases}$$

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$$

$$\begin{array}{ll} x & u \\ y & v \\ & w \end{array}$$

eg $\omega(x, y, z) = \sqrt{x^2+y^2+z^2}$

$x = 3e^t \sin s$

$y = 3e^t \cos s$

$z = 4e^t$

What is $\frac{\partial \omega}{\partial t}$ at $s = t = 0$?

(sol) $\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial t}$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot 3e^t \sin s + \frac{y}{\sqrt{x^2+y^2+z^2}} \cdot (+3e^t \cos s)$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot 4e^t$$

at $s=t=0$, $(x, y, z) = (0, 3, 4)$

$$= \frac{0}{5} \cdot 0 + \frac{3}{5} \cdot 3 + \frac{4}{5} \cdot 4$$

$$= \frac{9+16}{5} = 5.$$

eg P is moving with position at time t

by $\begin{cases} x(t) = t^2+1 \\ y(t) = 2t^2 \end{cases}$ where altitude is

given by $H(x, y) = x^2 - y^2 + 100$

① Is P going up/down at $t=1$?

② Which direction should P moves at $t=1$ to go down most quickly?

(sol) ① we need to find $\left.\frac{\partial H}{\partial t}\right|_{t=1}$.

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$x = t^3+1$

$y = 2t^2$

$$= (2x)\cdot(3t^2) + (-2y)\cdot(4t)$$

$$= 2(t^3+1)(3t^2) + (-2\cdot 2t^2)(4t)$$

$$= 6t^5 + 6t^2 - 16t^3$$

$$\therefore \left.\frac{\partial H}{\partial t}\right|_{t=1} = 6+6-16 = -4 < 0.$$

$\therefore$ P is moving down at $t=1$.

② Recall that f decrease most rapidly in the direction of $-\nabla f$.

At $t=1$, $(x,y) = (2,2)$

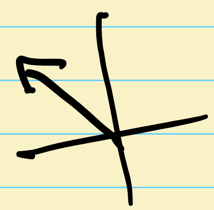
$\nabla H = (2x, -2y)$

$\nabla H(2,2) = (4,-4)$

$\therefore$ At $t=1$, H decreases most rapidly in the direction $-\nabla H(2,2) = (-4,4)$

$\parallel$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



P should moves toward $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Rmk

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial H}{\partial y} \cdot \frac{\partial y}{dt}$$

slope in x-direction

velocity in x-dir

slope in y-direction

velocity in y-dir.

$$= \nabla H \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}_{\text{velocity}}$$

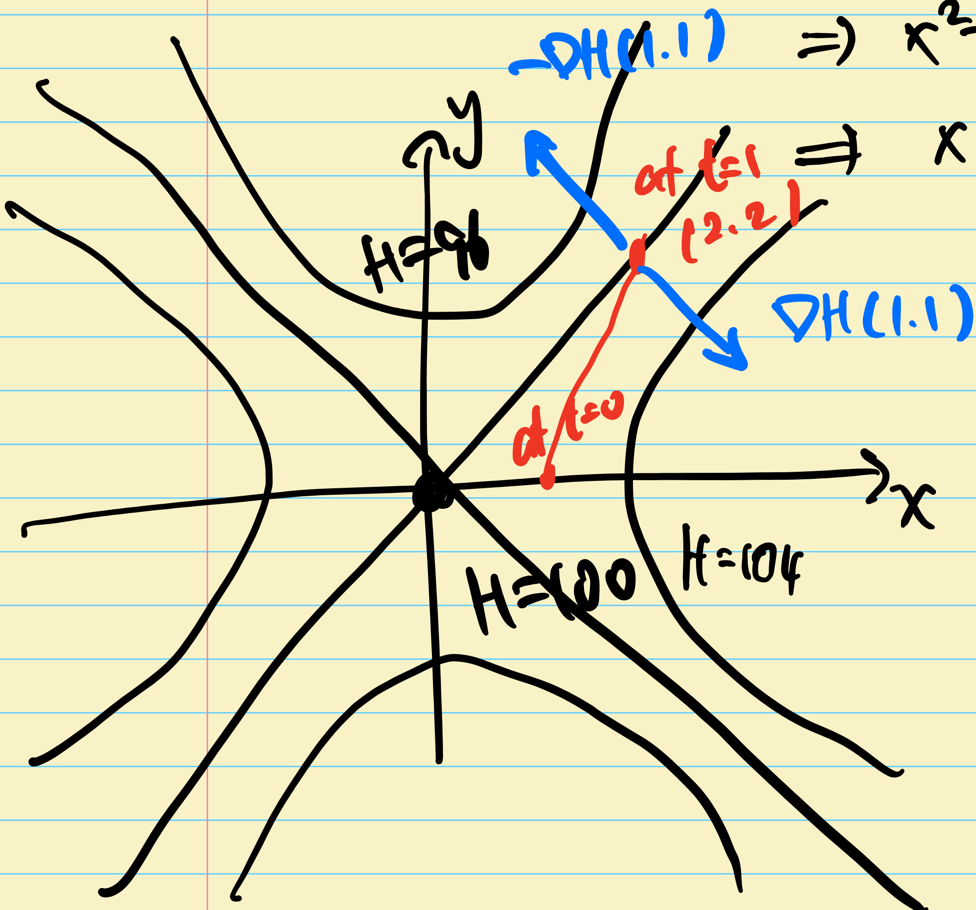
Picture

level set of $H(x,y) = x^2 - y^2 + 100.$

level set at $100 \Rightarrow x^2 - y^2 + 100 = 100$

$\Rightarrow x^2 - y^2 = 0$

$\Rightarrow x = y$ or $x = -y$



level set at 104

$x^2 - y^2 + 100 = 104$

$x^2 - y^2 = 4$